INITIAL-EQUILIBRIUM-STATE DEPENDENCE OF DAY-TO-DAY DYNAMICS UNDER BOUNDED RATIONALITY

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Introduction

Day-to-day dynamics of network flow evolution are to model how the network flow pattern evolves after some network disruption or facility change takes place. The studies on day-to-day dynamics can be roughly categorized into two classes, deterministic models and stochastic models. Deterministic day-to-day models all provides explicit flow evolution trajectory (e.g., Smith, 1984; Friesz et al., 1994; Nagurney and Zhang, 1996; Yang, 2005). Stochastic day-to-day models may focus on the probability distribution of flow states and/or the expected flow state (e.g., Cascetta, 1989; Davis and Nihan, 1993; Hazelton and Watling, 2004). There are also works combing (or studying both) stochastic and deterministic models (e.g., Canterella and Cascetta, 1995; Yang and Liu, 2007). Several studies used the day-to-day dynamical system approach to study the stability of network equilibrium (e.g. Horowitz, 1984; Watling, 1999; Bie and Lo, 2010). He et al. (2010) pointed out that many earlier path-based deterministic day-to-day models have two shortcomings, namely the path-flow-nonuniqueness problem and the path-overlapping problem. They proposed a link-based model to overcome the two problems.

Recently travelers' bounded rationality in route choice has been incorporated into day-to-day dynamic models (Guo and Liu, 2011). The concept of bounded rationality has been extensively studied in the economic and psychology literature, and it has been shown that bounded rationality is important in many contexts (see, e.g., Conlisk, 1996). In transportation field, there are only a small number of studies on bounded rationality. Mahmassani and Chang (1987) studied the existence, uniqueness, and stability properties of BRUE in the standard single-link bottleneck network. Many simulation and experimental studies have incorporated travelers' boundedly rational behaviors (e.g., Jayakrishnan et al., 1994; Hu and Mahamssani, 1997; Mahamssani and Liu, 1999; Mahamssani 2000). The simulation results of Nakayama et al. (2001) imply a need to examine the validity of the perfect rationality assumption in traffic equilibrium analysis. Szeto and Lo (2006) used the bounded rationality formulation in their dynamic traffic assignment problem. Lou et al. (2010) is the first to systematically examine the mathematical properties of BRUE in a network traffic assignment context. They provided some basic formulation and concepts of the BRUE flow solution, e.g., nonuniqueness and non-convexity of the BRUE flow set.

Under a boundedly rational (BR) day-to-day dynamic, travelers' perception errors are allowed to vary within a presumed bound, and the network flow pattern starting from a disequilibrium state will evolve towards a boundedly rational user equilibrium (BRUE) state rather than the traditional Wardrop user equilibrium (UE) state (Guo and Liu, 2011). Because BRUE is a set of equilibrium flow states (rather than a single equilibrium flow state as UE), BR day-to-day dynamics can model irreversible network changes. That is, when a network change is revoked and the network flow pattern changes back, it only changes back to the original BRUE flow set, not necessarily back to the original BRUE flow point.

This paper demonstrates that a BR day-to-day dynamic has the initial-equilibrium-state dependence property. That is, simply given a starting disequilibrium state, a BR day-to-day dynamic cannot be used to predict future flow evolutions. The equilibrium state (i.e., BRUE state) that precedes the disequilibrium state must also be given. This property has significant implications on BR day-to-day dynamic modeling. Specifically, if travelers' bounded rationality in route choice is to be captured in a day-to-day dynamic, the impact of the initial equilibrium condition has to be explicitly considered. This means that many existing day-to-day dynamic modeling methods, where the next day's flow pattern depends only on the current day's situation (and thereby a starting disequilibrium state uniquely determines a flow evolution trajectory), cannot be directly extended to the case under bounded rationality.

Day-to-Day Dynamics under Bounded Rationality

Let a transportation network be a fully-connected directed graph denoted as $G(N,L)$, consisting of a set of nodes N and a set of links L. Let W be the set of OD pairs, d_w be the fixed travel demand between OD pair $w \in W$, P_w be the set of paths connecting OD pair $w \in W$, f_{pw}^{t} be the path flow on path $p \in P_w$ on day t, x_a^t be the link flow on link $a \in L$ on day t. Denote demand, path flow and link flow vectors as $\mathbf{d}$, $\mathbf{f}^t$, and $\mathbf{x}^t$, respectively. Let $\mathbf{A}$ be the link-path incidence matrix, then $\mathbf{x}^t = \mathbf{A}\mathbf{f}^t$. Let $\mathbf{\Phi}$ be the OD-path incidence matrix, then $\mathbf{d\Phi f}^{t}$. Let $c_a(\mathbf{x})$ be the link cost function of link $a \in L$, then $c_a(\mathbf{x}^t)$ is the link cost of link $a \in L$ on day t, and we denote $\mathbf{c}(\mathbf{x}^t)$ as the corresponding link cost vector. Let $\mathbf{F}^t$ denote the path cost vector on day t, with individual path cost F_{pw}^t, then it holds $\mathbf{F}^t = \mathbf{A}'\mathbf{c}(\mathbf{x}^t) = \mathbf{A}'\mathbf{c}(\mathbf{A}\mathbf{f}^t)$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

The above notations are sufficient for describing discrete-time day-to-day traffic dynamics. For continuous-time versions, we denote the day-to-day path flow dynamic as $\dot{\mathbf{f}}$, which is the derivative of path flow with respect to time, and denote the day-to-day link flow dynamic as $\dot{\mathbf{x}}$. It holds readily $\dot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{f}}$.

Denote the feasible path flow set as $\Omega_f = \left\{ \mathbf{f} : \mathbf{\Phi f} = \mathbf{d}, \mathbf{f} \geq 0 \right\}$, and the feasible link flow set as $\Omega_x = \left\{ \mathbf{x} : \mathbf{x} = \mathbf{Af}, \mathbf{f} \in \Omega_f \right\}$. We give a formal definition of BRUE as below.

Definition 1. A path flow pattern $\mathbf{f} \in \Omega_f$ is said to be a boundedly rational user equilibrium (BRUE) flow pattern if it holds that

$$F_{pw} \leq \mu_w + \varepsilon_w \text{, if } f_{pw} > 0 \text{, } w \in W \tag{1}$$

where μ_w is the shortest path cost between OD pair $w \in W$ under flow $\mathbf{f}$, and $\varepsilon_w \geq 0$ is the bounded rationality threshold of travelers between OD pair $w \in W$.

In the above definition, condition (1) simply states that, under a BRUE flow pattern, the travel cost of any used path can be higher than the shortest path, but within a threshold. Observe that, when the bounded rationality threshold is zero, i.e., $\varepsilon_w = 0$ for all $w \in W$, condition (1) reduces to $F_{pw} = \mu_w$ for all used paths, and thus the BRUE definition becomes the classic UE definition. Also note that, the UE flow pattern always satisfies condition (1) (due to $F_{pw} = \mu_w$ on all used paths), and thus is always one BRUE solution.

In some simulation studies (e.g., Hu and Mahamssani, 1997), the bounded rationality threshold ε_w is given as a percentage of the minimum OD cost μ_w rather than a constant value. For example, $\varepsilon_w = 0.1\mu_w$ means that the cost of any used path at BRUE should be not more than 10% higher than the minimum OD cost.

The BRUE link flow definition is given as follows.

Definition 2. A link flow pattern $\mathbf{x} \in \Omega_x$ is said to be a BRUE link flow pattern if there exists a BRUE path flow $\mathbf{f} \in \Omega_f$ such that $\mathbf{x} = \mathbf{Af}$.

Note that the above BRUE link flow definition does not put any mathematical restriction on the flow pattern beyond the original BRUE definition.

A BR day-to-day dynamic is one such that the network flow pattern starting from a disequilibrium state (non-BRUE state) will evolve towards a BRUE state. Guo and Liu (2011) gave a link-based BR day-to-day dynamic to model the phenomenon of irreversible network change.

Initial-Equilibrium-State Dependence of BR Day-to-Day Dynamics

In this section we demonstrate that the network flow evolution under bounded rationality depends not only on the starting disequilibrium flow state, but also on the initial equilibrium flow state.

We use the same illustrative network as in He et al. (2010). Consider a simple network shown in Figure 1, consisting of 4 nodes and 5 links with shown node and link numbers. There is one OD pair from Node O to Node D connected by four paths numbered as below:

Path 1, link sequence $1 \rightarrow 3 \rightarrow 4$,
Path 2, link sequence $1 \rightarrow 3 \rightarrow 5$,
Path 3, link sequence $2 \rightarrow 3 \rightarrow 4$,
Path 4, link sequence $2 \rightarrow 3 \rightarrow 5$.

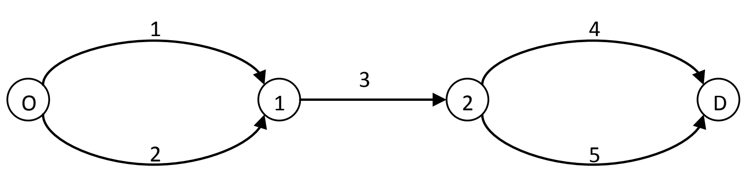

Figure 1. A network to demonstrate the initial-equilibrium-state dependence of BR day-to-day dynamics

Consider the flowing link cost functions

$$\begin{cases} c_1(x_1) = 20 + 5x_1 \\ c_2(x_2) = 24 + x_2 \\ c_3(x_3) = 2 + x_3 \\ c_4(x_4) = 20 + 5x_4 \\ c_5(x_5) = 24 + x_5 \end{cases} \tag{2}$$

Consider that the travel demand between Node O and Node D is $d = 3$, and the flow state on day 0 is

$$\mathbf{x}^0 = (x_1, x_2, x_3, x_4, x_5)' = (2,1,3,2,1)'$$

which, based on link cost functions (2), gives a link cost vector

$$\mathbf{c}(\mathbf{x}^0) = (c_1, c_2, c_3, c_4, c_5)' = (30, 25, 5, 30, 25)'$$

and a path cost vector

$$\mathbf{F}^0 = (F_1, F_2, F_3, F_4)' = (65, 60, 60, 55)'$$

Observe that the path flow vector $\mathbf{f}^0$ corresponding to link flow vector $\mathbf{x}^0$ is not unique but captured by the following linear equation system

$$\begin{cases} f_1 + f_2 = x_1 = 2 \\ f_1 + f_3 = x_4 = 2 \\ f_2 + f_4 = x_5 = 1 \\ f_3 + f_4 = x_2 = 1 \end{cases} \tag{3}$$

From linear equation system (3) it can be verified that it holds $f_1 \geq 1$ under given link flow vector $\mathbf{x}^0$.

Consider that the bounded rationality threshold parameter is $\varepsilon = 6$, which is roughly 10% of the average OD travel time 60. Then it can be easily seen that the above flow state $\mathbf{x}^0$ is a disequilibrium state (non-BRUE state): Path 1 carries a flow $f_1 \geq 1$ while the cost of Path 1 is higher than the minimum OD cost with a difference 10, exceeding the bounded rationality threshold $\varepsilon = 6$. Because the flow state $\mathbf{x}^0$ on day 0 is a disequilibrium state, the network flow pattern will evolve towards an equilibrium state (BRUE state) as time goes on. We can apply a deterministic BR day-to-day dynamic (e.g., Guo and Liu 2011) to model the flow evolution process and predict the BRUE state that will be attained.

The network topology is characterized by two subnetworks, the subnetwork from Node O to Node 1 and the subnetwork from Node 2 to Node D. Because the starting flow state $\mathbf{x}^0$ is symmetric with respect to the two subnetworks, the cost structure given by link cost functions (2) is also symmetric with respect to the two subnetworks, and all day-to-day dynamics are based on flow and cost conditions of the network, it is expected that the flow evolution starting from $\mathbf{x}^0$ will also be symmetric on both subnetworks.

So far it seems that the application of a BR dynamic has nothing different than the application of any traditional deterministic day-to-day dynamics: given a starting disequilibrium state $\mathbf{x}^0$, a deterministic day-to-day dynamic model can predict a unique flow evolution trajectory. However, as to be shown below, the incorporation of bounded rationality in day-to-day dynamics makes the starting disequilibrium state $\mathbf{x}^0$ alone insufficient in predicting future flow evolutions.

Consider the following two scenarios.

Link-1 Lane Closure Scenario: the original Link 1 cost function was

$$\tilde{c}_1(x_1) = 20 + 2x_1,$$

other links had the same cost functions as given by (2), and the network was originally at flow state $\mathbf{x} = \mathbf{x}^0$ with link cost vector

$$\mathbf{c}(\mathbf{x}) = (c_1, c_2, c_3, c_4, c_5)' = (24, 25, 5, 30, 25)'$$

and path cost vector

$$\mathbf{F} = (F_1, F_2, F_3, F_4)' = (59, 54, 60, 55)'$$

It can be verified from the above path cost vector that this original flow state $\mathbf{x} = \mathbf{x}^0$ was a BRUE flow (maximum path cost difference not exceeding the BR threshold $\varepsilon = 6$) and thus could be regarded as a long term equilibrium flow pattern. On day 0, a lane was closed on Link 1 so that the Link 1 cost function became the one given by (2). Then the flow state $\mathbf{x} = \mathbf{x}^0$ was no longer BRUE as mentioned earlier and the flow pattern will evolve starting from $\mathbf{x} = \mathbf{x}^0$.

Link-4 Lane Closure Scenario: the original Link 4 cost function was

$$\tilde{c}_4(x_4) = 20 + 2x_4,$$

other links had the same cost functions as given by (2), and the network was originally at flow state $\mathbf{x} = \mathbf{x}^0$ with link cost vector

$$\mathbf{c}(\mathbf{x}) = (c_1, c_2, c_3, c_4, c_5)' = (30, 25, 5, 24, 25)'$$

and path cost vector

$$\mathbf{F} = (F_1, F_2, F_3, F_4)' = (59, 60, 24, 55)'$$

It can be verified from the above path cost vector that this original flow state $\mathbf{x} = \mathbf{x}^0$ is a BRUE flow (maximum path cost difference not exceeding the BR threshold $\varepsilon = 6$) and thus could be regarded as a long term equilibrium flow pattern. On day 0, a lane was closed on Link 4 so that the Link 4 cost function became the one given by (2). Then the flow state $\mathbf{x} = \mathbf{x}^0$ was no longer BRUE as mentioned earlier and the flow pattern will evolve starting from $\mathbf{x} = \mathbf{x}^0$.

In both the Link-1 and the Link-4 lane closure scenarios, after the lane closures happened on day 0, the new network conditions in terms of the cost structure and the starting disequilibrium state are exactly the same. Therefore, applying any traditional deterministic day-to-day dynamics, the flow evolution processes in the two lane-closure scenarios will be the same, which, however, is problematic in view of the following observations.

Similar to He et al. (2010), by examining the network shown in Figure 1, we can see that the network is “separable”: the subnetwork from Node O to Node 1 and the subnetwork from Node 2 to Node D are totally independent of each other. As a result, a lane closure on Link 1 should not affect the flow split between Link 4 and 5, and a lane closure on Link 4 should not affect the flow split between Link 1 and 2. More rigorously, it could be stated as below:

Observation 1. *For the network shown in Figure 1, assuming a fixed travel demand and separable link cost functions (i.e. no spillback effect), and consider that the network flow is originally at stable equilibrium, then, if a lane closure on Link 1 takes place, the flow split between Link 4 and Link 5 should remain stable and unchanged. If a lane closure on Link 4 takes place, the flow split between Link 1 and Link 2 should remain stable and unchanged*.

Observation 1 is a reasonable and logical “expectation” about the network shown in Figure 1, and a model that violates this expectation is at least not amenable to this small network. According to Observation 1, in the Link-1 lane closure scenario, the disequilibrium flow evolution process should occur only on Link 1 and 2, while in the Link-4 lane closure scenario, the disequilibrium flow evolution process should occur only on Link 4 and 5. On the other hand, as discussed earlier, when we apply any traditional day-to-day dynamic model, the two scenarios will give the same flow evolution process which will impact all the links.

The above contradiction implies that, simply given a starting disequilibrium state, a BR day-to-day dynamic cannot be used to predict future flow evolutions. The equilibrium state (BRUE state) that precedes the disequilibrium state must also be given. In the above example, the flow $\mathbf{x}^0$ on day 0 is not a BRUE flow and thus we know the network flow pattern will have to evolve. However, without knowing the original equilibrium scenario from which the current disequilibrium state $\mathbf{x}^0$ originates, we cannot tell how the network flow pattern will evolve. In summary, for a given disequilibrium state, there exists multiple initial equilibrium states (BRUE states) that could have lead to the disequilibrium state, without proper information on the initial equilibrium states, a BR day-to-day dynamic cannot be applied. This property of BR day-to-day dynamics is referred to as initial-equilibrium-state dependence in this paper.

Conclusions

This paper demonstrates the initial-equilibrium-state dependence property of day-to-day dynamic models under bounded rationality. That is, simply given a starting disequilibrium state, a BR day-to-day dynamic cannot be used to predict future flow evolutions. The equilibrium state (i.e., BRUE state) that precedes the disequilibrium state must also be given. This property has significant implications on BR day-to-day dynamic modeling. Specifically, if travelers' bounded rationality in route choice is to be captured in a day-to-day dynamic, the impact of the initial equilibrium condition has to be explicitly considered. This means that many existing day-to-day dynamic modeling methods, where the next day's flow pattern depends only on the current day's situation (and thereby a starting disequilibrium state uniquely determines a flow evolution trajectory), cannot be directly extended to the case under bounded rationality.

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