SEQUENTIAL TOLL IMPLEMENTATION TO GUIDE NETWORK FLOW EVOLUTION UNDER BOUNDED RATIONALITY

Xiaolei Guo, University of Windsor

Introduction

Traditional transportation network modeling studies typically assume that travelers are perfectly rational, i.e., they always choose the paths with the shortest (perceived or real) travel costs. In reality, however, users are boundedly rational in the sense that they may choose non-shortest paths if the travel time saving offered by switching to the shortest path is not big enough. The concept of bounded rationality (BR) has been extensively studied in the economic and psychology literature, and it has been shown that BR is important in many contexts (see, e.g., Conlisk, 1996).

In transportation field, there are only a small number of studies on BR. Mahmassani and Chang (1987) studied the existence, uniqueness, and stability properties of boundedly rational user equilibrium (BRUE) in the standard single-link bottleneck network. Many simulation and experimental studies have incorporated travelers' boundedly rational behaviors (e.g., Jayakrishnan et al., 1994; Hu and Mahamssani, 1997; Mahamssani and Liu, 1999; Mahamssani 2000). The simulation results of Nakayama et al. (2001) imply a need to examine the validity of the perfect rationality assumption in traffic equilibrium analysis. Lou et al. (2010) is the first to systematically examine the mathematical properties of BRUE in a network traffic assignment context. Guo and Liu (2011) developed a boundedly rational day-to-day dynamic model to explain irreversible network change. Zhang (2012) discussed and compared four equilibrium traffic assignment models including BRUE and pointed out that the general BR theory must be further developed and refined for traffic assignment modeling.

One important property of BRUE is that the BRUE equilibrium solution is not unique, i.e., BRUE always has a set of solutions instead of a unique one (e.g., Lou et al., 2010). This nonuniqueness property can make traditional network planning and design strategies ineffective. For example, when marginal cost congestion pricing is implemented with the aim of achieving the system optimal (SO) flow pattern, the worst-case BRUE network performance can be even worse than that under the no toll case. Generally speaking, because of the nonuniqueness of BRUE, from a static equilibrium perspective, if we admit/accept that travelers are boundedly rational, then there is no guarantee of attainability of any target flow pattern by implementing tolls or other network design strategies. Lou et al. (2010) tackles this problem by introducing robust congestion pricing which considers worst-case and best-case solutions under BRUE.

In this paper, we solve the nonuniqueness problem of BRUE from a disequilibrium flow evolution perspective. Based on two assumptions on travelers' route choice behaviors which are consistent with BRUE, we design toll sequence operations (TS-operations) which can induce the network flow pattern to evolve towards the traditional Wardrop user equilibrium (UE) flow pattern. In particular, under homogeneous BR, iteratively implementing our TS-operations can make the network flow pattern converge to UE. This essentially solves the nonuniqueness problem of BRUE and re-establishes the effectiveness of link tolls in realizing any target link flow pattern. That is, given a target link flow pattern (e.g., the SO flow), we can always implement a link toll scheme under which the target link flow is a tolled UE flow (Bai et al. 2006; Guo and Yang, 2009), and then we can iteratively implement our TS-operations to realize this tolled UE flow.

Preliminaries on Homogenous Bounded Rationality

Let a transportation network be a fully-connected directed graph denoted as $G(N,L)$, consisting of a set of nodes N and a set of links L. Let W be the set of OD pairs, d_w be the fixed travel demand between OD pair $w \in W$, P_w be the set of paths connecting OD pair $w \in W$, and $P = \bigcup_{w \in W} P_w$ be the set of all paths of the network. Paths are assumed to be acyclic. Let f_p be the path flow on path $p \in P$, and x_a be the link flow on link $a \in L$. The following relationships and constraints hold

$$x_a = \sum_{p \in P} f_p \delta_{ap}, \ a \in L \tag{1}$$

$$d_w = \sum_{p \in P_w} f_p, \ w \in W \tag{2}$$

$$f_p \geq 0, \ p \in P \tag{3}$$

where δ_{ap} is equal to 1 if path p uses link a and 0 otherwise. Denote demand, path flow and link flow vectors as $\mathbf{d}$, $\mathbf{f}$, and $\mathbf{x}$, respectively, then the feasible set of link flows is given by $X = \{\mathbf{x} : \text{there exists an } \mathbf{f} \text{ such that (1)-(3) hold}\}$, and the feasible set of path flows is given by $\Omega = \{\mathbf{f} : \text{constraints (2)-(3) hold}\}$.

In this paper, we consider separable link travel time function $c_a(x_a)$, $a \in L$, which means that the travel time of one link depends on the flow on the link only. It is assumed that $c_a(x_a)$ is an increasing function of x_a for all $a \in L$. Let C_p be the travel time along path $p \in P$, which is the sum of travel times on all links that constitute the path. We thus have

$$C_p = \sum_{a \in L} c_a(x_a) \delta_{ap}, \ p \in P \tag{4}$$

We recall now the BRUE definition (e.g., Guo and Liu, 2011).

Definition 1. A path flow pattern $\mathbf{f} \in \Omega$ is said to be a boundedly rational user equilibrium (BRUE) flow pattern if it holds that

$$C_p \leq \mu_w + \varepsilon_w, \text{ if } f_p > 0, \ \forall \ p \in P_w, \ w \in W \tag{5}$$

where μ_w is the shortest path cost between OD pair $w \in W$ under flow $\mathbf{f}$, and $\varepsilon_w \geq 0$ is the BR threshold of travelers between OD pair $w \in W$.

In the above definition, condition (5) simply states that, under a BRUE flow pattern, the travel cost of any used path can be higher than the shortest path, but within a threshold. Observe that, when the BR threshold is zero, i.e., $\varepsilon_w = 0$ for all $w \in W$, condition (5) reduces to $C_p = \mu_w$ for every used path $p \in P_w$, and thus the BRUE definition becomes the classic UE definition. Also note that, the UE flow pattern always satisfies condition (5) (due to $C_p = \mu_w$ on every used path $p \in P_w$), and thus there is always one BRUE solution.

Unlike UE, the BRUE equilibrium solution is not unique, i.e., BRUE always has a set of solutions instead of a unique one (Lou et al., 2010). This means that, from a static equilibrium perspective, there is no guarantee of attainability of any target flow pattern (first-best or second-best) by implementing tolls or other network design strategies. In the following we will design TS-operations that can induce the network flow pattern to evolve towards the traditional UE pattern, which essentially solves the nonuniqueness problem of BRUE and re-establishes the effectiveness of link tolls in realizing any target link flow pattern.

Because our scheme is built upon network flow evolution, we need to make assumptions regarding how travelers change their routes when the network flow pattern is at some disequilibrium state (non-BRUE state). One most natural assumption in this regard would be one implied by the definition of BRUE, that a traveler would change to a shorter route only if the route is more than certain threshold shorter than her current route.

To formally state the above assumption and technically deal with flow evolution, let $\mathbf{f}(t)$ and $\mathbf{x}(t)$ be the path flow and link flow vectors, respectively, on day t, and let $\dot{\mathbf{f}}$ and $\dot{\mathbf{x}}$ be the derivatives of path flow and link flow, respectively, with respect to time t. Furthermore, let $\dot{f}_{p\to q}$ be the net flow swapping rate from path p to path q, and define $\dot{f}_{q\to p} = -\dot{f}_{p\to q}$ between any two paths p and q. That is, to say that travelers are switching from path p to path q at a positive rate $\dot{f}_{p\to q} > 0$, is equivalent to say that travelers are switching from path q to path p at a negative rate $\dot{f}_{q\to p} = -\dot{f}_{p\to q} < 0$. Simply speaking, $\dot{f}_{p\to q}$ and $\dot{f}_{q\to p}$ describes the same flow swapping process between path p and path q. Then we have

$$\dot{f}_p = -\sum_{q\in P} \dot{f}_{p\to q} = \sum_{q\in P} \dot{f}_{q\to p} \tag{6}$$

which means that the flow changing rate of a path is equal to the sum of all the inflow rates towards the path, or equivalently, is equal to the negative of the sum of all the outflow rates from the path. We use $p \sim q$ to denote that paths p and q connects the same OD pair, then $\dot{f}_{p\to q} \neq 0$ can hold only if $p \sim q$.

Because we are going to use tolls to induce flow evolution, we introduce toll related notations here. We only consider link toll in this paper. Let τ_a be the toll charged on link $a \in L$, and let $\boldsymbol{\tau}$ be the link toll vector. Let g_p be the generalized cost of path $p \in P$, which is the sum of travel times and tolls on all links that constitute the path. We have

$$g_p = \sum_{a\in L} c_a\left(x_a\right)\delta_{ap} + \sum_{a\in L} \tau_a \delta_{ap} = C_p + \sum_{a\in L} \tau_a \delta_{ap}\,,\ p \in P \tag{7}$$

Now we are ready to formally state our first assumption regarding flow evolution dynamics.

Assumption 1. A traveler may switch to a new route only if the new route is more than certain threshold shorter than her current route. Mathematically, it can hold $\dot{f}_{p\to q} > 0$ only if $p \sim q$, $f_p > 0$ and $g_p > g_q + \varepsilon$, $\forall\ \ p \in P$, $q \in P$.

Note that Assumption 1 essentially assumes homogeneous users with the same level of BR threshold across the network. This homogeneous BR assumption will be used in this paper, and thus we will rewrite ε_w to be ε .

With Assumption 1, if we let $\varepsilon_w = \varepsilon$ in eqn. (5) of the BRUE Definition 1 and consider the no toll case $\boldsymbol{\tau} = 0$, then it is clear that at BRUE it holds $\dot{f}_{p \to q} = 0$ for all p and q, and therefore it holds $\dot{\mathbf{f}} = 0$, i.e., a BRUE flow pattern is a stationary flow pattern under Assumption 1. Note that Assumption 1 alone does not exclude other flow patterns to be stationary, i.e., BRUE is sufficient yet not necessary for a stationary flow. To make BRUE a necessary and sufficient condition for stationary flow, we further introduce the following assumption.

Assumption 2. Some travelers on a path must switch to other paths if there exists a path that is more than certain threshold shorter than their current path. Mathematically, for any path $p \in P$, if $f_p > 0$ and there exists a path $r \sim p$ such that $g_p > g_r + \varepsilon$, then $\dot{f}_{p \to q} > 0$ must hold for at least one path $q \sim p$.

Comparing Assumptions 1 and 2, we can see that Assumption 1 specifies a condition under which a flow *can* change, while Assumption 2 gives a condition under which a flow *must* change. Combining Assumptions 1 and 2, and consider the no toll case $\boldsymbol{\tau} = 0$, we can see that a flow is a stationary flow if and only if it is a BRUE flow.

To discuss our TS-operations later, here we first define an ε -toll.

Definition 2. A toll $\boldsymbol{\tau}$ is said to be an ε -toll if under toll $\boldsymbol{\tau}$ it holds

$$C_p \le g_p \le C_p + \varepsilon \,, \ p \in P \tag{8}$$

Comparing (8) and (7), if we only consider non-negative tolls, then (8) simply means that the total toll along any path should be not higher than ε . For example, if we only levy a toll $0 \le \tau_a \le \varepsilon$ on link

$a \in L$ and let the tolls on all other links be 0, then obviously we obtain an ε -toll. As we will see later, this simplest kind of ε -toll actually plays an important role in our main task of realizing UE flow pattern.

Effective ε -toll-sequence operation to realize the UE flow

In the following, we are going to show that, if a flow pattern is not UE (say, a BRUE flow that is not UE), then levy an ε -toll can induce the flow to move towards UE, and by doing so iteratively, the flow pattern can be finally led to UE.

To do so, let us revisit the UE objective function. Because in this paper we look at flow evolution from a path flow swapping perspective, we write the UE objective function in terms of path flow variables

$$Z(\mathbf{f}) = \sum_{a \in L} \int_0^{x_a} c_a(\omega) d\omega = \sum_{a \in L} \int_0^{\sum_{p \in P} \delta_{ap} f_p} c_a(\omega) d\omega \qquad (9)$$

It is well known that the UE path flow pattern, or equivalently the path flow pattern $\mathbf{f} \in \Omega$ that minimizes objective function (9), is typically not unique. We let $\mathbf{\Omega}^{\text{UE}}$ denote the set of UE path flow patterns.

It is easy to derive that the gradient vector of objective function (9) is just the path cost vector, i.e., $\nabla Z(\mathbf{f}) = \mathbf{C}$, where $\mathbf{C}$ is the path cost vector. Then taking time derivative of (9) gives

$$\frac{dZ(\mathbf{f}(t))}{dt} = \left(\nabla Z(\mathbf{f})\right)^T \dot{\mathbf{f}} = \sum_{p \in P} C_p \dot{f}_p \qquad (10)$$

Substituting eqn. (6) into eqn. (10) gives

$$\frac{dZ(\mathbf{f}(t))}{dt} = \sum_{p \in P} \sum_{q \in P} C_p \dot{f}_{q \to p} = -\sum_{p \in P} \sum_{q \in P} C_p \dot{f}_{p \to q}$$

which readily gives

$$\frac{dZ(\mathbf{f}(t))}{dt} = \frac{1}{2}\left(\sum_{p \in P} \sum_{q \in P} C_p \dot{f}_{q \to p} - \sum_{p \in P} \sum_{q \in P} C_p \dot{f}_{p \to q} \right) \qquad (11)$$

By simple index rearranging, we have

$$\sum_{p\in P}\sum_{q\in P} C_p \dot{f}_{p\to q} = \sum_{q\in P}\sum_{p\in P} C_q \dot{f}_{q\to p} = \sum_{p\in P}\sum_{q\in P} C_q \dot{f}_{q\to p}$$

Thus, by rewriting the last term of the right-hand side of eqn. (11), we have

$$\begin{aligned}\frac{dZ\left(\mathbf{f}(t)\right)}{dt} &= \frac{1}{2}\left(\sum_{p\in P}\sum_{q\in P} C_p \dot{f}_{q\to p} - \sum_{p\in P}\sum_{q\in P} C_q \dot{f}_{q\to q}\right) \\ &= \frac{1}{2}\sum_{p\in P}\sum_{q\in P}\left(C_p - C_q\right)\dot{f}_{q\to p}\end{aligned} \tag{12}$$

Now we are ready to give the following important lemma.

Lemma 1. With Assumption 1, if an ε -toll is implemented, it holds

$$\frac{dZ\left(\mathbf{f}(t)\right)}{dt} \le 0$$

where "=" holds if and only if $\dot{f}_{p\to q} = 0$ for all paths p and q.

Proof: From (12), it suffices to prove that, for any pair of paths $p \sim q$, $\dot{f}_{q\to p} > 0 \Rightarrow C_q > C_p$. Without loss of generality, consider a pair of paths $p \sim q$ such that $\dot{f}_{q\to p} > 0$, then from Assumption 1 we have

$$g_q > g_p + \varepsilon \tag{13}$$

Because the toll is an ε -toll, from eqn. (8) of Definition 2 we have

$$C_q + \varepsilon \ge g_q \tag{14}$$

$$g_p \ge C_p \tag{15}$$

Combining (13) and (14) gives $C_q > g_p$, and then from (15) we have $C_q > C_p$. This completes the proof.

The above proof follows a simple line: eqn. (12) guarantees that if flow can only change from a higher cost path (cost not including toll) to a lower cost one, then the UE objective function will reduce if flow changes; Assumption 1 guarantees that, under an ε -toll, flow can only change from a higher cost path (cost not including toll) to a lower cost one. The key part here is that, under an ε -toll, the total toll along any path is not higher than ε , which, together with Assumption 1, guarantees that no one will be forced to swap to a longer path by tolls.

From Lemma 1, if an ε -toll is implemented and the flow pattern changes under the toll, then the UE objective function (9) reduces. We formally state this as Lemma 2.

Lemma 2. With Assumption 1, if an ε -toll is implemented on an initial flow $\mathbf{f} \in \Omega$ for a time period of $t > 0$ and the resulting flow is $\mathbf{f}'$, then it holds $Z(\mathbf{f}') \leq Z(\mathbf{f})$, where "=" holds if and only if $\mathbf{f}' = \mathbf{f}$.

From Lemma 2, if we can design a sequence of ε -tolls, under which the flow pattern will keep changing until UE is achieved, then such a toll sequence can be implemented to guide flow evolution towards the UE flow pattern.

To this end, let us formally introduce the concept of a TS-operation. Let $\Gamma = \{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_n\}$ be a sequence of n tolls and $S = \{t_1, t_2, ..., t_n\}$ be a sequence of n positive time periods, we define $\Psi = (\Gamma, S)$ as a TS-operation, where toll $\boldsymbol{\tau}_i$ is implemented for a time period of t_i and followed by toll $\boldsymbol{\tau}_{i+1}$ being implemented for a time period of t_{i+1}, $i = 1, 2, ..., n-1$. Note that it can hold $\boldsymbol{\tau}_i = 0$, which represents a no-toll period of t_i. For a given TS-operation $\Psi = (\Gamma, S)$, let $\Psi(\mathbf{f})$ denote the flow obtained by implementing Ψ on an initial flow $\mathbf{f} \in \Omega$. Assuming the flow evolution dynamic satisfies the technical uniqueness conditions for differential systems, then $\Psi(\mathbf{f})$ could be viewed as a continuous function of $\mathbf{f}$, i.e., $\Psi: \ \Omega \rightarrow \Omega$.

Definition 3. A TS-operation $\Psi = (\Gamma, S)$ is said to be an ε *-TS-operation* if the toll sequence Γ consists of ε -tolls only.

Furthermore, an ε -TS-operation $\boldsymbol{\Psi}$ is said to be *effective* if $\Psi(\mathbf{f}) \neq \mathbf{f}$ holds for any $\mathbf{f} \in \Omega \setminus \Omega^{\text{UE}}$.

We will discuss and give examples of effective ε -TS-operations later. Here we first give the following important result regarding an effective ε -TS-operation.

Lemma 3. With Assumption 1, if $\Psi = (\Gamma, S)$ is an effective ε -TS-operation, then it holds $Z(\Psi(\mathbf{f})) < Z(\mathbf{f})$ for any $\mathbf{f} \in \Omega \setminus \Omega^{\text{UE}}$.

Observe that Lemma 3 is a direct result of Lemma 2. Lemma 3 simply states that implementing an effective ε -TS-operation on a non-UE flow pattern will guide the flow towards UE.

Define $\Psi^n(\mathbf{f}) = \Psi(\Psi^{n-1}(\mathbf{f}))$ for all integers $n \geq 2$, i.e., $\Psi^n(\mathbf{f})$ is the flow obtained by iteratively implementing $\boldsymbol{\Psi}$ for n times starting from initial flow $\mathbf{f} \in \Omega$. From Lemma 3, applying Zangwill's convergence theorem (Zangwill 1969), we have the following general theorem of realizing the UE flow pattern.

Theorem 1. With Assumption 1, if there exists an effective ε -TS-operation $\Psi = (\Gamma, S)$, then, starting from any initial flow $\mathbf{f} \in \Omega$, either $\Psi^n(\mathbf{f}) \in \Omega^{\text{UE}}$ holds for a finite integer n, or it holds $\Psi^n(\mathbf{f}) \to \mathbf{f}^*$, $\mathbf{f}^* \in \Omega^{\text{UE}}$, as $n \to \infty$.

Theorem 1 states that implementing an effective ε -TS-operation iteratively will make the flow pattern converge to UE, or can realize a flow that is arbitrarily close to the UE flow pattern.

Now the question becomes whether there exists an effective ε -TS-operation. We shall start by giving the following result, which ensures the existence of an effective ε -TS-operation.

Theorem 2. Let a_1, a_2,..., $a_{|L|}$ be an ordering of the elements of L, where $|L|$ is the cardinality of L, i.e., the number of links of the network. Let $\Gamma = \{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_{|L|}\}$ be a toll sequence where $\boldsymbol{\tau}_i$ is such that $\tau_{a_i} = \varepsilon$ and all other links have 0 toll, i.e., $\boldsymbol{\tau}_i = (0, 0, ..., \tau_{a_i} = \varepsilon, ..., 0)^T$, $i = 1, 2, ..., |L|$. Let $S = \{t_1, t_2, ..., t_{|L|}\}$ be a sequence of time periods where $t_i > 0$, $i = 1, 2, ..., |L|$. Then, with Assumptions 1 and 2, the TS-operation $\Psi = (\Gamma, S)$ is an effective ε-TS-operation.

Theorem 2 states that implementing a toll equal to ε on one link at a time and doing so over all links of the network is an effective ε-TS-operation, which, from Theorem 1, can guide the flow towards the UE flow pattern if implemented iteratively. It should be mentioned that Theorem 2 here is just to show the existence of an effective ε-TS-operation, while the particular TS-operation presented in Theorem 2 is not necessarily the best. In particular, an effective ε-TS-operation does not necessarily require toll implementation on every link. In the following, we will show that a subset of links could be enough for constructing an effective ε-TS-operation.

Definition 4. A link set $A \subseteq L$ is said to be effective if for any $\mathbf{f} \in \Omega \setminus \Omega^{\mathrm{UE}}$, for at least one pair of paths $p \sim q$, $f_p > 0$ and $C_p > C_q$ (such a pair of paths must exist, otherwise $\mathbf{f} \in \Omega^{\mathrm{UE}}$), there exists a link $a \in A$ such that $\delta_{ap} = 1$ and $\delta_{aq} = 0$.

Observe that there always exists an effective link set because L itself is effective. For an effective link set, we have the following result.

Theorem 3. Consider an effective link set $A \subseteq L$, let a_1, a_2,..., $a_{|A|}$ be an ordering of the elements of A, where $|A|$ is the cardinality of

A . Let $\Gamma = \left\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ..., \boldsymbol{\tau}_{|A|}\right\}$ be a toll sequence where $\boldsymbol{\tau}_i$ is such that $\tau_{a_i} = \varepsilon$ and all other links have 0 toll, i.e., $\boldsymbol{\tau}_i = \left(0, 0, ..., \tau_{a_i} = \varepsilon, ..., 0\right)^T$, $i = 1, 2, ..., |A|$. Let $S = \left\{t_1, t_2, ..., t_{|A|}\right\}$ be a sequence of time periods where $t_i > 0$, $i = 1, 2, ..., |A|$. Then, with Assumptions 1 and 2, the TS-operation $\Psi = (\Gamma, S)$ is an effective ε -TS-operation.

Theorem 3 simply states that implementing a toll equal to ε on one link at a time and doing so over all links of an effective link set is an effective ε -TS-operation.

Strong cut-set to construct effective link set

We are naturally interested in finding a minimal effective link set, which involves a minimum number of links when implementing an effective ε -TS-operation. In the following we shall show that, by examining the strong cut-sets of OD pairs, we can obtain effective link sets which involves fewer number of links.

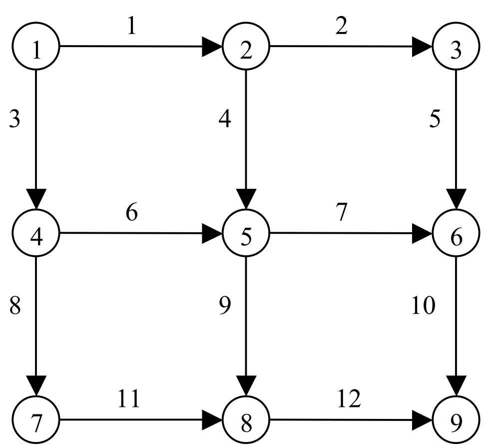

Figure 1. A small network to illustrate strong cut-set

Let $CS_w \subseteq L$ denote a *strong cut-set* of OD pair $w \in W$ such that for every path $p \in P_w$ there exists *one and only one* link $a \in CS_w$ such that $\delta_{ap} = 1$. Because we only consider simple paths without cycles, a strong cut-set always exists for an OD pair. For example, the set of all links starting from the origin node is a strong cut-set, and the set of all links ending at the destination node is also a strong cut-set. In the small network shown in Figure 1, node and link numbered as shown, if we consider OD pair from Node 1 to Node 9 (OD $1 \rightarrow 9$), then Links 1 and 3 constitute a strong cut-set, and so do Links 10 and 12. other strong cut-sets of OD $1 \rightarrow 9$ in this small example, e.g., the set of Links 3, 4 and 5, the set of Links 9, 10 and 11, etc.

A strong cut-set, in spite of comprising a relatively limited number of links, involves all paths, and thus is likely to be an effective link set. That is, sequentially implementing a toll equal to ε on all links of a strong cut-set can impact all paths and thus could be an effective ε-TS-operation. There are two exceptions we need to take care of. First, if a strong cut-set consists of one link only, then a toll on this link is equally added to every path and thus will not impact travelers' route choice. Second, for a strong cut-set that consists of two or more links, when all travelers choose the same link of the strong cut-set, we may have the same problem as the first exception. To handle the first exception, we need to consider strong cut-sets comprised of two or more links; to handle the second one, we define a balanced strong cut-set.

Definition 5. A strong cut-set $CS_w \subseteq L$ of OD pair $w \in W$ is said to be balanced if $|CS_w| \geq 2$, and, when there exists a link $a \in CS_w$ such that $\delta_{ap} = 1$ holds for every path $p \in P_w$, $f_p > 0$, there must exist a path $q \in P_w$ such that $\delta_{aq} = 0$, $C_q < C_r$ for a path $r \in P_w$, $f_r > 0$.

Definition 5 means that, for a balanced strong cut-set CS_w of OD pair $w \in W$, if all travelers between OD pair $w \in W$ use the same one link of CS_w, then there must exists one (unused) path that passes through another link of CS_w with a path cost lower than one used

path. In other words, if a strong cut-set CS_w is balanced, then there does not exist such a situation that all travelers use one link of CS_w, while all the (unused) paths passing through other links of CS_w are longer than the used paths. For the small network shown in Figure 1, consider the strong cut-set of OD $1 \to 9$, Links 1 and 3. If Links 1 and 3 constitutes a balanced strong cut-set, then, if all travelers use Link 1, then at least one unused path passing through Link 3 must be shorter than one used path.
Balanced strong cut-sets have the following property.

Theorem 4. Consider a link set $A = \bigcup_{w \in W} CS_w$, where $CS_w \subseteq L$ is a balanced strong cut-set of OD pair $w \in W$, then A is an effective link set.

Theorem 4 basically gives a heuristic about finding an effective link set. That is, we find a balanced strong cut-set for every OD pair, then the union of these strong cut-sets is an effect link set. For the small network shown in Figure 1, consider there is only one OD pair $1 \to 9$ and the strong cut-set comprised of Links 1 and 3 is balanced, then from Theorems 1, 3 and 4, implementing a toll equal to ε on Links 1 and 3 alternately and iteratively will guide the flow towards UE.

Conclusions

We end this paper by summarizing the findings. Theorem 1 states that iteratively implementing an effective ε-TS-operation will make the flow pattern converge to UE, and Theorem 2 states that an effective ε-TS-operation always exists. Theorems 3 and 4 highlight that an effective ε-TS-operation does not necessarily involves all links, and in particular, strong cut-sets of OD pairs can be used to construct effective link sets. These theorems together guarantee that the UE flow pattern can be realized under homogeneous BR, which essentially solves the nonuniqueness problem of BRUE and re-establishes the effectiveness of link tolls in realizing any target link flow pattern. That is, given a target link flow pattern, we can always implement a link toll scheme under which the target link flow is a tolled UE flow (Bai et al. 2006; Guo and Yang, 2009), and then we can iteratively implement an effective ε -TS-operation to realize this tolled UE flow.

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