

# **PARETO-IMPROVING CONGESTION PRICING AND REVENUE REFUNDING WITH ELASTIC DEMAND**

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## **Introduction**

Guo and Yang (2010) studied Pareto-improving congestion pricing cum revenue refunding (CPRR) schemes with heterogamous users, which make every road user better off as compared with the situation without congestion pricing. Their work adopted a fixed demand model. In this paper we study CPRR schemes with elastic demand. Practically, revenue refunding with elastic demand could be implemented in the form of the credit-based congestion pricing strategy (Kalmanje and Kockelman, 2004; Kockelman and Kalmanje, 2005). Adopting an analytical approach, this study provides theoretical answers to the following questions: is it necessary to refund all the toll revenue to travelers for making everyone better off? If not, how much revenue reserve can be retained by the government, and otherwise could be used to cover the operation cost of CPRR schemes? What are the main determinants of the amount of revenue reserve that would be generated?

We tackle demand elasticity by treating auto trip demand between each origin-destination (OD) pair as a function of full trip price (or

generalized travel cost by car) in a conventional manner, which captures user heterogeneity in reservation price for making a auto trip and/or choice of driving frequency due to different travel budgets. Nevertheless, travelers are treated to be identical in their trade-off between money and travel time saving, this setback of identical value of time (VOT) is unavoidable in order to make our analysis analytically tractable.

The remainder of the paper is organized as follows. We begin our analysis with the simplest case, a single OD pair connected by a single congested link, to illustrate our basic idea. We then extend our analysis to a general network traffic equilibrium model with elastic demand. Finally, some concluding remarks are presented.

### Single link network with elastic demand

Consider a single road connecting a single OD pair. Simple as it is, this scenario could apply to a cordon-based road pricing scheme. For example, the cordon-based pricing scheme in Durham is essentially a charge on a single link. Let  $Q$  denote the auto travel demand between the OD pair,  $B(Q)$  be the inverse demand function (the marginal benefit function), and  $t(Q)$  be the link travel time function. We assume that  $B(Q)$  is monotonically decreasing, and  $t(Q)$  is monotonically increasing. Let  $\tilde{Q}$  be the equilibrium travel demand in the absence of toll charge, we have

$$B(\tilde{Q}) = t(\tilde{Q}) \quad (1)$$

Let  $\bar{Q}$  be the equilibrium travel demand when a toll (in equivalent time)  $u > 0$  is introduced, we have

$$B(\bar{Q}) = t(\bar{Q}) + u \quad (2)$$

Because  $B(Q)$  is decreasing and  $t(Q)$  is increasing, we have  $\bar{Q} < \tilde{Q}$  due to  $u > 0$ , thus a number,  $(\tilde{Q} - \bar{Q})$ , of trips are priced out of driving by the toll charge.

We need to make the following assumptions for our subsequent analyses to be valid.

**Assumption 1.** *A traveler's pre-toll incentive to driving is not distorted by the future refund.*

**Assumption 2.** *A traveler's post-toll driving decision under pricing is not influenced by the refund.*

Assumption 1 states that a CBRR scheme to be implemented in the future does not distort travelers' current travel decisions in the pre-toll situation (choice of driving and driving frequency if driving). This assumption is made to ensure that the untolled equilibrium is always given by (1) regardless of the receipt of future refunds or the types of the future CPRR scheme. Assumption 2 states that a refunding scheme should not alter travelers' decision of whether or not to drive or drive how much under pricing. This will ensure that the tolled equilibrium should be determined by the toll level as given by (2), which is independent of the post-toll refunds. We will show how these two assumptions will be fulfilled through a careful design of the CPRR schemes.

We now design a revenue refunding scheme which can make every traveler better off compared with the "do-nothing" case. For easy presentation, travelers are ordered in their decreasing traveler benefit per auto trip, i.e. the  $q$ -th traveler's trip benefit is  $B(q)$ . We assume, for easy presentation, that each traveler, if choosing to drive, drives every day during the modeling period considered. In this case, the  $q$ -th traveler is equivalently referred to as "trip  $q$ " on a particular day.

A traveler is said to be a *revealed* traveler if she travels by car in the absence of toll charge, and a *latent* traveler otherwise. Namely, the  $q$ -th traveler is revealed if  $q \leq \tilde{Q}$ , and is latent if  $q > \tilde{Q}$ .

To ensure that Assumption 2 is always satisfied, we consider a uniform rebate per pre-toll revealed traveler no matter whether she

continues or stops driving after introduction of the CPRR scheme. Let  $\phi$  be such a uniform amount of refund to every pre-toll revealed traveler. With toll  $u$  and refund  $\phi$ , each traveler solves a discrete choice problem of “to drive or not to drive”. For a revealed traveler  $q$ ,  $q \leq \tilde{Q}$ , if she chooses to drive, her net benefit, denoted by  $\hat{B}(q)$ , is

$$\hat{B}(q) = \phi + B(q) - (t(\bar{Q}) + u) \quad (3)$$

and if she chooses not to drive, her net benefit is

$$\hat{B}(q) = \phi \quad (4)$$

The traveler’s choice depends on the relative magnitude of the net benefit given by (3) and (4). Clearly, traveler  $q$  will choose to drive if  $B(q) \geq (t(\bar{Q}) + u)$ , and not to drive otherwise, which is as if there is no refund. In other words, the uniform refund does not alter the post-toll driving decision of the pre-toll revealed travelers.

The above analysis only considers the pre-toll revealed travelers; while the pre-toll latent travelers have to be considered with the CPRR scheme. In this case, the refund level should be limited so that the pre-toll latent travelers are not induced to drive by the post-toll refunds. If a latent traveler chooses to drive under pricing, her net benefit is given by (3), assuming that she receives a share of refund. If she chooses to stay uninvolved, her net benefit is zero as she receives no refund. Thus, to keep the latent travelers uninvolved, the amount of refund,  $\phi$ , is required to satisfy

$$\phi + B(q) - (t(\bar{Q}) + u) < 0, \text{ for any } q > \tilde{Q}$$

which is equivalent to

$$\phi < (t(\bar{Q}) + u) - B(q), \text{ for any } q > \tilde{Q} \quad (5)$$

Because  $B(q)$  is a decreasing function, the right-hand side of (5) increases with  $q$ , thus setting  $q = \tilde{Q}$  in (5) gives the following critical condition

$$\phi \leq (t(\bar{Q}) + u) - B(\tilde{Q}) \quad (6)$$

In view of the equilibrium conditions (1) and (2), condition (6) can be rewritten as

$$\phi \leq (t(\bar{Q}) + u) - t(\tilde{Q}) = B(\bar{Q}) - B(\tilde{Q}) \quad (7)$$

Condition (7) means that the amount of refund should not be larger than the increase in car trip disutility brought about by toll charge; otherwise some of the pre-toll latent travelers will be attracted to drive.

Now we move on to look into the possibility of making every revealed traveler better off through revenue refunding. In the absence of toll charge, the net benefit,  $\hat{B}(q)$ , of a revealed traveler  $q$ ,  $q \leq \tilde{Q}$ , is

$$\hat{B}(q) = B(q) - B(\tilde{Q}), \quad q \leq \tilde{Q} \quad (8)$$

With toll  $u$  and refund  $\phi$ , the net benefit becomes

$$\hat{B}(q) = \begin{cases} \phi + B(q) - B(\bar{Q}), & q \leq \bar{Q} \\ \phi, & \bar{Q} < q \leq \tilde{Q} \end{cases} \quad (9)$$

Comparing (8) and (9), the change in net benefit is

$$\Delta \hat{B}(q) = \begin{cases} \phi - (B(\bar{Q}) - B(\tilde{Q})), & q \leq \bar{Q} \\ \phi - (B(q) - B(\tilde{Q})), & \bar{Q} < q \leq \tilde{Q} \end{cases} \quad (10)$$

We require that the change in net benefit of *every* revealed traveler is nonnegative. In view of the fact that  $B(q) < B(\bar{Q})$  for  $\bar{Q} < q \leq \tilde{Q}$  in eqn. (10), The critical condition for every revealed traveler being made better off is

$$\phi \geq B(\bar{Q}) - B(\tilde{Q}) = (t(\bar{Q}) + u) - t(\tilde{Q}) \quad (11)$$

Condition (11) means that the amount of refund should not be less than the increase in auto trip disutility; otherwise some of the revealed travelers will be made worse off as compared with the “do-nothing” case.

To sum up, for a refund  $\phi$  to keep the latent travelers uninvolved, condition (7) must hold, and to make every revealed traveler better off, condition (11) must hold. These two conditions together imply that the following per driver per pre-toll car trip based refunding scheme

$$\phi = B(\bar{Q}) - B(\tilde{Q}) = (t(\bar{Q}) + u) - t(\tilde{Q}) \quad (12)$$

Consequently, in order to make sure that the refund does not change the auto trip demand and make every revealed traveler better off (or more rigorously, not worse off), we should refund uniformly to each revealed traveler an amount equal to the driving disutility increase brought about by the toll charge. We shall refer to the refunding scheme,  $\phi = B(\bar{Q}) - B(\tilde{Q})$ , as *the per pre-toll-trip based Pareto-improving refunding scheme*.

To see the effect of the Pareto-improving refunding scheme on different groups of travelers, we substitute  $\phi = B(\bar{Q}) - B(\tilde{Q})$  into (10) and obtain the following change in net benefit of each revealed traveler

$$\Delta \hat{B}(q) = \begin{cases} 0, & q \leq \bar{Q} \\ B(\bar{Q}) - B(q), & \bar{Q} < q \leq \tilde{Q} \end{cases} \quad (13)$$

Clearly, with the Pareto-improving refunding scheme, the net benefits of those who continue driving after pricing do not change, and those who stop driving after pricing enjoy positive increase in their net benefits.

It should be particularly mentioned here that the Pareto-improving revenue refunding scheme  $\phi = B(\bar{Q}) - B(\tilde{Q})$  does not require any information on the demand function, because  $B(\bar{Q}) = t(\bar{Q}) + u$  and  $B(\tilde{Q}) = t(\tilde{Q})$  are the observed auto travel disutilities at equilibrium with and without toll charge, respectively. Thus our analysis applies to realistic situations where demand functions are unknown.

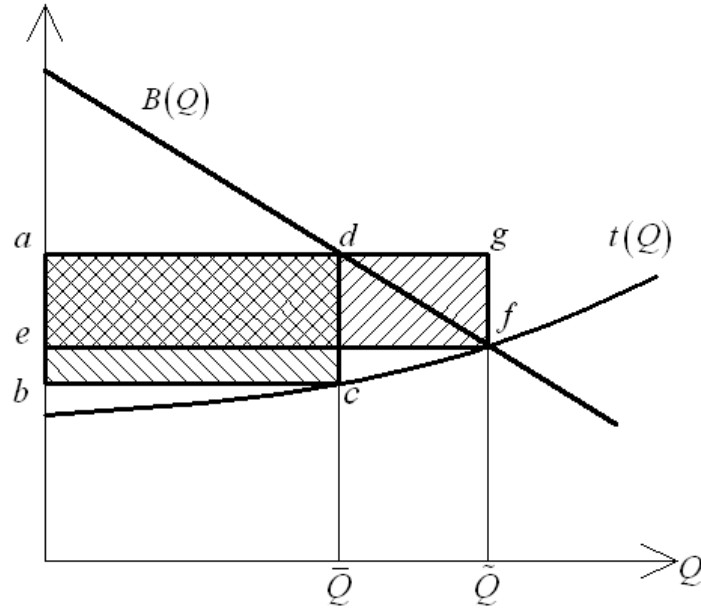
### Budget constraint of refunding by toll revenue

In this subsection we move on to check under what conditions the Pareto-improving refunding scheme  $\phi = B(\bar{Q}) - B(\tilde{Q})$  is within the budget, i.e., the total amount of refund does not exceed the total toll revenue collected. Mathematically, we need

$$\phi\tilde{Q} \leq u\bar{Q} \quad (14)$$

It should be noted that, because  $\tilde{Q} > \bar{Q}$ , condition (14) is not trivial. This can be easily seen from Figure 1. In this Figure, the area of rectangular  $\overline{abcd}$  represents the total toll revenue  $u\bar{Q}$ , and the area of rectangular  $\overline{aefg}$  represents the total amount of refund  $\phi\tilde{Q}$ . Thus the budget constraint  $\phi\tilde{Q} \leq u\bar{Q}$  graphically means that the area  $\overline{aefg}$  is smaller than the area  $\overline{abcd}$ , which, as can be seen in the Figure, does not obviously hold.

There is one more point here on the budget constraint. If a toll increases the social welfare, then we know that the total toll revenue collected (area  $\overline{abcd}$  in Figure 1) is larger than the consumer surplus loss (area of trapezoid  $\overline{aefd}$ ). This means that, if we refund the revenue to each individual user according to her net benefit loss, then the total revenue is enough to make everyone better off. However,  $\phi = B(\bar{Q}) - B(\tilde{Q})$  is a *uniform* rather than an individual-specific refunding scheme, which is required for keeping the traffic demand unchanged. Under the proposed uniform refunding scheme, the group of users who stop driving after pricing are indeed over refunded, and the total amount of refund (area  $\overline{aefg}$ ) is larger than the consumer surplus loss (area  $\overline{aefd}$ ). Thus, even if it is known that the social welfare is increased, it is still unclear whether the total amount of refund required (area  $\overline{aefg}$ ) is larger or smaller than the total toll revenue (area  $\overline{abcd}$ ).



**Figure 1.** Graphical representation of the budget constraint

We shall show that the Pareto-improving refunding scheme does not violate the budget constraint as long as the toll level does not exceed the system optimal level (the marginal cost pricing toll) or it does not over depress the travel demand. To do so, denote  $T(Q) = t(Q)Q$ , which is the total system travel time,  $M(Q) = dT(Q)/dQ$ , which is the marginal cost function, and let  $Q^*$  be the demand level at system optimum (SO). We have

$$B(Q^*) = M(Q^*) \quad (15)$$

Equation (15) simply represents the well-known result that, at SO, the marginal benefit is equal to the marginal cost, or the inverse demand curve intersects the marginal cost curve at SO. It is also well-known that, when there is congestion effect, i.e.  $t(Q)$  increases with  $Q$ , it



holds that  $Q^* < \tilde{Q}$ , namely, the SO traffic volume is lower than the untolled equilibrium volume.

**Theorem 1<sup>1</sup>.** *Assume that  $T(Q) = t(Q)Q$  is strictly convex in  $Q$ . If the tolled demand level  $\bar{Q}$  meets  $Q^* \leq \bar{Q} < \tilde{Q}$ , then it holds that  $\phi\tilde{Q} < u\bar{Q}$  for the Pareto-improving refunding scheme  $\phi = B(\bar{Q}) - B(\tilde{Q})$ .*

Theorem 1 states that, if the system travel time function is strictly convex, then the total toll revenue would not be used up by the Pareto-improving refunding scheme as long as the travel demand is not over depressed by the toll charge. Here “over depressed” means that the demand level under toll is even lower than the SO level, which corresponds to a toll higher than the SO toll level. Note that  $T(Q)$  is strictly convex as long as  $t(Q)$  is strictly increasing and convex.

### General networks with elastic demand

In this section we extend our results to a general road network. A transportation network is considered as a fully-connected directed graph denoted as  $G(N, A)$ , consisting of a set of nodes  $N$  and a set of links  $A$ . Denote the set of OD pairs as  $W$ , the travel demand for OD pair  $w \in W$  as  $d_w$ , the set of paths connecting OD pair  $w \in W$  as  $R_w$ , the flow on path  $r \in R_w$  as  $f_{rw}$ , the flow on link  $a \in A$  as  $v_a$ , the toll charged on link  $a \in A$  as  $u_a$ , and the link-path incidence matrix as  $\Delta$ . Let  $\mathbf{d}$ ,  $\mathbf{f}$ ,  $\mathbf{v}$  and  $\mathbf{u}$  be the respective column vectors, then  $\mathbf{v}'\mathbf{u}$  is the total toll revenue collected. Let  $\Omega(\mathbf{d})$  denote the feasible link flow set with demand  $\mathbf{d} \geq 0$ :

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<sup>1</sup> Due to space limit, proofs of all theorems are omitted. Full proofs are available from authors upon request.

$$\Omega(\mathbf{d}) = \left\{ \mathbf{v} : \mathbf{v} = \Delta \mathbf{f}, \mathbf{f} \geq 0, \sum_{r \in R_w} f_{rw} = d_w, w \in W \right\} \quad (16)$$

Our analysis does not rely on separable link travel time functions, thus we consider general link travel time functions, with  $t_a(\mathbf{v})$  being the link travel time function of link  $a \in A$ , and  $\mathbf{t}(\mathbf{v})$  being the link travel time vector. To guarantee the uniqueness of traffic equilibrium, we assume that  $\mathbf{t}(\mathbf{v})$  is strictly monotonic.

Our analysis relies on separable auto trip demand functions. Let  $B_w(d_w)$  be the inverse demand function (marginal benefit function) of OD pair  $w \in W$ . Let  $\tilde{\mathbf{d}}$  and  $\tilde{\mathbf{v}}$  be the equilibrium demand and link flow vectors in the absence of toll charge,  $\bar{\mathbf{d}}$  and  $\bar{\mathbf{v}}$  be the equilibrium demand and link flow vectors when a toll  $\mathbf{u}$  is implemented. Following the analysis in the single-link case, we know that an amount  $\phi_w = B_w(\bar{d}_w) - B_w(\tilde{d}_w)$  evenly refunded to each pre-toll revealed traveler of OD pair  $w \in W$  would make every revealed traveler better off, while keep the tolled equilibrium demand unchanged. In vector form, this Pareto-improving refunding scheme is  $\Phi = \mathbf{B}(\bar{\mathbf{d}}) - \mathbf{B}(\tilde{\mathbf{d}})$ , and  $\tilde{\mathbf{d}}' \Phi$  is the total amount of refund.

We shall check the budget constraint  $\tilde{\mathbf{d}}' \Phi \leq \bar{\mathbf{v}}' \mathbf{u}$  for the Pareto-improving refunding scheme  $\Phi = \mathbf{B}(\bar{\mathbf{d}}) - \mathbf{B}(\tilde{\mathbf{d}})$ . We first need to decompose the two effects of congestion pricing in a general network, i.e., demand depression to optimize demand level, and user rerouting to optimize traffic assignment. To this end, we define the following optimal value function:

$$\Gamma(\mathbf{d}) = \min_{\mathbf{v} \in \Omega(\mathbf{d})} T(\mathbf{v}), \mathbf{d} \geq 0 \quad (17)$$

Clearly,  $\Gamma(\mathbf{d})$  is the minimum total system travel time associated with demand  $\mathbf{d} \geq 0$ . With the assumption of strictly convex  $T(\mathbf{v})$ , it can be shown that  $\Gamma(\mathbf{d})$  is strictly convex and continuously

differentiable, and its gradient vector  $\nabla\Gamma(\mathbf{d})$  is just the vector of OD marginal system cost at the corresponding SO with given demand  $\mathbf{d}$ .

Now we move on to examine the budget constraint of the Pareto-improving CPRR scheme. We first consider the SO solution  $(\mathbf{v}^*, \mathbf{d}^*)$ , which solves the following SO problem:

$$\min_{\mathbf{d} \geq 0, \mathbf{v} \in \Omega(\mathbf{d})} T(\mathbf{v}) - \sum_{w \in W} \int_0^{d_w} B_w(z) dz \quad (18)$$

The SO problem (18) in a minimization problem form is equivalent to maximizing the social welfare. With  $\Gamma(\mathbf{d})$  defined by (17), it holds readily for the SO solution  $(\mathbf{v}^*, \mathbf{d}^*)$  that

$$T(\mathbf{v}^*) = \Gamma(\mathbf{d}^*) \quad (19)$$

Namely,  $T(\mathbf{v}^*)$  is the minimum total system travel time under demand  $\mathbf{d}^*$ . With (19), by rewriting the SO problem (18), we know that  $\mathbf{d}^*$  solves the following problem:

$$\min_{\mathbf{d} \geq 0} \Gamma(\mathbf{d}) - \sum_{w \in W} \int_0^{d_w} B_w(z) dz \quad (20)$$

The optimality condition of problem (20) gives

$$\left( \nabla\Gamma(\mathbf{d}^*) - \mathbf{B}(\mathbf{d}^*) \right)' (\mathbf{d} - \mathbf{d}^*) \geq 0 \text{ for any } \mathbf{d} \geq 0 \quad (21)$$

where  $\nabla\Gamma(\mathbf{d}^*)$  is the gradient vector of  $\Gamma(\mathbf{d})$  at  $\mathbf{d} = \mathbf{d}^*$ , and  $\left( \nabla\Gamma(\mathbf{d}^*) - \mathbf{B}(\mathbf{d}^*) \right)$  is the gradient vector of the objective function (20) at  $\mathbf{d} = \mathbf{d}^*$ . Condition (21) means that, deviating from the SO demand  $\mathbf{d} = \mathbf{d}^*$  along any feasible direction  $(\mathbf{d} - \mathbf{d}^*)$  will not further decrease the objective value (20) (not increase the social welfare). If we consider an interior SO solution  $\mathbf{d}^* > 0$ , then condition (21) becomes

$$\nabla\Gamma(\mathbf{d}^*) = \mathbf{B}(\mathbf{d}^*) \quad (22)$$

Condition (22) is the counterpart of condition (15) in a general network, which simply means that, at SO, the marginal social cost is equal to the marginal benefit for each OD pair (with positive demand).

**Theorem 2.** *Assume that  $T(\mathbf{v})$  is strictly convex in  $\mathbf{v}$ . If a toll  $\mathbf{u}$  realizes the SO solution  $(\mathbf{v}^*, \mathbf{d}^*)$  and  $\mathbf{d}^* \neq \tilde{\mathbf{d}}$ , then it holds  $\tilde{\mathbf{d}}' \Phi < \mathbf{u}' \mathbf{v}^*$  for the Pareto-improving refunding scheme  $\Phi = \mathbf{B}(\mathbf{d}^*) - \mathbf{B}(\tilde{\mathbf{d}})$ . Furthermore, we have*

$$\mathbf{u}' \mathbf{v}^* - \tilde{\mathbf{d}}' \Phi = \Delta \tilde{T} + \gamma^* \quad (23)$$

where  $\Delta \tilde{T} = T(\tilde{\mathbf{v}}) - \Gamma(\tilde{\mathbf{d}}) \geq 0$

and  $\gamma^* = \Gamma(\tilde{\mathbf{d}}) - \Gamma(\mathbf{d}^*) - \mathbf{B}(\mathbf{d}^*)' (\tilde{\mathbf{d}} - \mathbf{d}^*) > 0$ .

Theorem 2 states that, under the assumption of strictly convex total system travel time function, the Pareto-improving CPRR scheme does not use up the total toll revenue if the toll system realizes the SO demand and link flows. Furthermore, the resulting *revenue reserve* consists of two parts,  $\Delta \tilde{T}$  and  $\gamma^*$ . Clearly,  $\Delta \tilde{T} = T(\tilde{\mathbf{v}}) - \Gamma(\tilde{\mathbf{d}})$  is the efficiency loss caused by socially inefficient route choice at the untolled traffic equilibrium, associated with the given demand  $\tilde{\mathbf{d}}$ . In other words,  $\Delta \tilde{T}$  is the efficiency gain that can be achieved by the socially optimal route choice for given travel demand  $\tilde{\mathbf{d}}$ . Thus  $\Delta \tilde{T}$  reflects the user rerouting effect of congestion pricing in a general network. On the other hand,  $\gamma^*$  is determined by the demand levels  $\tilde{\mathbf{d}}$  and  $\mathbf{d}^*$ , and the properties of the functions  $\Gamma(\mathbf{d})$  and  $\mathbf{B}(\mathbf{d})$ . Thus  $\gamma^*$  reflects the demand depression effect of road pricing. To sum up, the revenue reserve of an SO toll after refunding, i.e. the total SO toll revenue subtracted by the total amount of Pareto-improving refund, consist of two parts, one representing the socially optimal user rerouting effect and the other reflecting the socially optimal demand depression effect brought about by the pricing system.

For the single-link network studied in previous section, it is clear that  $\Delta\tilde{T}=0$ , i.e. road pricing has no user rerouting effect, thus the revenue reserve is just  $\gamma^*$  resulting from demand depression. In contrast, it typically holds that  $\Delta\tilde{T}>0$  in a general network. Because the revenue reserve is strictly larger than  $\Delta\tilde{T}$ , the amount  $\Delta\tilde{T}$  can be regarded as a benchmark for CPRR in general networks. That is,  $\Delta\tilde{T}=T(\tilde{\mathbf{v}})-\Gamma(\tilde{\mathbf{d}})$  justifies an amount of revenue that the government can retain after implementing revenue redistribution, which can be estimated by simple fixed-demand analysis.

So far we have examined the Pareto-improving CPRR scheme for a toll pattern that achieves an SO solution (system optimal case). A strictly positive revenue reserve given in Theorem 2 implies that, for a second-best toll system that gives  $(\bar{\mathbf{v}}, \bar{\mathbf{d}})$  not too far away from the SO solution  $(\mathbf{v}^*, \mathbf{d}^*)$ , the Pareto-improving refunding scheme can still be within the budget by continuity. However, unlike the single-link case, where we can easily identify the condition that the demand should not be over-depressed, here the refunding budget constraint for a general network is somewhat complicated. We need to introduce the concept of a proper link toll pattern.

**Definition 1.** A link toll pattern  $\mathbf{u}$  is said to be a proper toll pattern if the equilibrium flow and demand  $(\bar{\mathbf{v}}, \bar{\mathbf{d}})$  under the toll charge satisfies

$$\Delta\bar{T}=T(\bar{\mathbf{v}})-\Gamma(\bar{\mathbf{d}})\leq\Delta\tilde{T} \quad (24)$$

$$(\nabla\Gamma(\bar{\mathbf{d}})-\mathbf{B}(\bar{\mathbf{d}}))'(\tilde{\mathbf{d}}-\bar{\mathbf{d}})\geq 0 \quad (25)$$

In the above definition,  $\Delta\bar{T}$  and  $\Delta\tilde{T}$ ,  $\Delta\tilde{T}=T(\tilde{\mathbf{v}})-\Gamma(\tilde{\mathbf{d}})$  defined before, are the efficiency losses caused by route choice at the tolled and untolled equilibrium, respectively, thus condition (24) means that, compared with the untolled situation, a proper link toll pattern

should not increase the efficiency loss caused by route choice. Note that  $(\nabla\Gamma(\bar{\mathbf{d}}) - \mathbf{B}(\bar{\mathbf{d}}))$  is the gradient vector of the objective function (20) at  $\mathbf{d} = \bar{\mathbf{d}}$ , thus condition (25) states that, deviating from the tolled demand  $\mathbf{d} = \bar{\mathbf{d}}$  along the direction  $(\tilde{\mathbf{d}} - \bar{\mathbf{d}})$  will not decrease the objective value (20) (not increase the social welfare). With strict convexity assumption of the objective function (20), condition (25) guarantees that the social welfare at  $\mathbf{d} = \bar{\mathbf{d}}$  is larger than that at  $\mathbf{d} = \tilde{\mathbf{d}}$ . Intuitively, condition (25) means that *the equilibrium demand  $\bar{\mathbf{d}}$  under a proper toll pattern should be somewhere between  $\tilde{\mathbf{d}}$  and  $\mathbf{d}^*$* . For the single-link case, a toll is proper if and only if it does not over depress the demand, which is stronger than that it increases the social welfare.

Now we can give the following theorem as a generalization of Theorem 2.

**Theorem 3.** *Assume that  $T(\mathbf{v})$  is strictly convex in  $\mathbf{v}$ . If a toll pattern  $\mathbf{u}$  is proper and gives equilibrium flow and demand  $(\bar{\mathbf{v}}, \bar{\mathbf{d}})$  such that  $\bar{\mathbf{d}} \neq \tilde{\mathbf{d}}$ , then it holds that  $\tilde{\mathbf{d}}' \Phi < \mathbf{u}' \bar{\mathbf{v}}$  for the Pareto-improving refunding scheme  $\Phi = \mathbf{B}(\bar{\mathbf{d}}) - \mathbf{B}(\tilde{\mathbf{d}})$ . Furthermore, we have*

$$\mathbf{u}' \bar{\mathbf{v}} - \tilde{\mathbf{d}}' \Phi = (\Delta\tilde{T} - \Delta\bar{T}) + \bar{\gamma} \quad (26)$$

where  $(\Delta\tilde{T} - \Delta\bar{T}) \geq 0$  and  $\bar{\gamma} = \Gamma(\tilde{\mathbf{d}}) - \Gamma(\bar{\mathbf{d}}) - \mathbf{B}(\bar{\mathbf{d}})'(\tilde{\mathbf{d}} - \bar{\mathbf{d}}) > 0$ .

Theorem 3 generalizes Theorem 2 into any second-best pricing situation with a proper tolling system, i.e. the Pareto-improving refunding scheme does not use up the total toll revenue if the toll system is proper, and the revenue reserve consists of two parts,  $(\Delta\tilde{T} - \Delta\bar{T})$  and  $\bar{\gamma}$ , the former representing the user rerouting effect and the latter reflecting the demand depression effect of the pricing system. Similar to the interpretation of Theorem 2, the term

$(\Delta\tilde{T} - \Delta\bar{T})$  justifies an amount of revenue that the government can retain after conducting revenue refunding, which can be estimated by simple fixed-demand analysis.

### **Conclusions**

We proposed a Pareto-improving CPRR scheme with elastic demand, which refunds to every pre-toll revealed traveler an amount of money equal to the travel disutility increase due to congestion pricing. The refunding scheme has no impact on travelers' post-toll trip-making decision, and thus does not alter the tolled equilibrium demand. The budget constraint of the Pareto-improving CPRR scheme was checked. For the single link case, under mild technical conditions, it was proved that the total toll revenue would not be used up by the Pareto-improving refunding scheme as long as the travel demand is not over depressed by the toll charge. For general networks, where congestion pricing has both demand depression and user rerouting effects, the Pareto-improving CPRR scheme does not exhaust the total toll revenue as long as the toll system is proper by our definition and the resulting revenue reserve consists of two parts pertaining to the two effects of congestion pricing. Because the user rerouting effect of congestion pricing involves fixed-demand analysis only, the corresponding part of revenue reserve is practically easy to estimate.

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